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Periodic solutions for a delayed neural network model on a special time scale[☆]

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ABSTRACT

This paper concerns a delayed neural network model

$$x^{\Delta}(t) = -\frac{1}{2}x(t) + f(x(t-2)), \quad t \in \mathbb{T} = \bigcup_{k=0}^{\infty}[2k, 2k+1].$$

Here $\mathbb{T}$ is a special time scale, and f is a signal transmission function. According to the discontinuity of signal function, by iteration, we obtain the existence of periodic solutions of the model and their asymptotical stability.

1. Introduction

Artificial neural networks can mimic the structure of the human brain and the functions of cells by electronic circuits. Hence the study of artificial neural networks has attracted the attention of many researchers. Most of the existing results are on the existence of equilibrium points and their stability and attractivity. See, for example, [1–4]. We should mention that, in the above cited references, the networks are described either by difference equations or by differential equations, that is, dynamical systems on $\mathbb{Z}$ or $\mathbb{R}$.

Recently, dynamical systems on time scales have gained much more attention because of two main reasons. One is that they can unify dynamical systems described by difference and differential equations. The study of dynamical systems on time scales goes back to its founder Stefan Hilger (1988), we refer the readers to [5]. The other reason is their potential for modeling the dynamics of neural networks, insect populations, epidemics and so on, see [6–9] and the references therein.

However, almost all results are theoretic and they are applicable to either $\mathbb{Z}$ or $\mathbb{R}$. As a result, in this paper, we consider the following dynamical system

$$x^{\Delta}(t) = -\frac{1}{2}x(t) + f(x(t-2)), \quad t \in \mathbb{T} = \bigcup_{k=0}^{\infty}[2k, 2k+1], \tag{1.1}$$

where $\mathbb{T}$ is a time scale that is unbounded above, which is neither discrete nor continuous, and $x^{\Delta}(t)$ is the (delta) derivative of x at t, which will be defined in Section 2. Eq. (1.1) arises as a delayed network model [10] for a single neuron, $-1/2$ is the

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internal decay rate, and $f : \mathbb{R} \to \mathbb{R}$ is a signal transmission function with McCulloch–Pitts nonlinearity and is defined by

$$f(x) = \begin{cases} -1 & \text{if } x > 0, \\ 1 & \text{if } x \le 0. \end{cases}$$

Since large scale networks are very hard to analyze, we use network (1.1) of one neuron as a prototype to understand the dynamics of large scale networks. Eq. (1.1) can also model the dynamics of insect populations that are continuous while in season (and many follow a difference scheme with variable step sizes) and die out in winter. During the period of disappearance, their eggs are incubating or dormant; after that the eggs hatch in a new season, giving rise to a nonoverlapping population. For more detail, we refer the readers to [6]. It is easy to see that, for any $\Phi : [0, 2]_{\mathbb{T}} \triangleq [0, 2] \cap \mathbb{T} \to \mathbb{R}$, (1.1) has a unique solution in $\mathbb{T}$ satisfying $x(t) = \Phi(t)$ for $t \in [0, 2]_{\mathbb{T}}$. We call such $\Phi(t)$ an initial value and the corresponding solution which satisfies the initial condition $x(t) = \Phi(t)$ for $t \in [0, 2]_{\mathbb{T}}$ is denoted by $x_\Phi(t)$.

In recent years, some authors have discussed the periodic solutions of neural network models, but they only discussed the difference and differential case [11–15]. Here, we consider a delayed neural network model on a special time scale, it may get some unexpected results. To our best knowledge, the question of periodic solutions of neural network model on time scales has not been considered. The purpose of this paper is to study the existence of periodic solutions of (1.1) and their asymptotical stability. The results demonstrate some particular dynamical behaviors from the discrete and continuous dynamical systems. The remaining part of this paper is organized as follows. In Section 2, we provide the basic terminology of time scales and some properties of solutions to (1.1). Then, in Section 3, we establish the main results of this paper, and numerical simulations are given at the end of the paper.

2. Preliminaries

In this section, we first state some definitions of time scales.

A *time scale* is a nonempty closed subset of $\mathbb{R}$, the set of all real numbers. $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, i.e., the integers, the positive integers, and the nonnegative integers are examples of time scales as well as $[0, 1] \cup [2, 3]$, $[0, 1] \cup \mathbb{N}$ and the Cantor set.

Let $\mathbb{T}$ be a time scale and $g : \mathbb{T} \to \mathbb{R}$. Fix $t \in \mathbb{T}$, define $f^\Delta(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that, for all $s \in U$,

$$|[g(\sigma(t)) - g(s)] - g^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|.$$

We call $g^\Delta(t)$ the (delta) derivative of g at t. The following statements are true.

(i) If t is right-scattered, then

$$g^\Delta(t) = \frac{g(\sigma(t)) - f(t)}{\mu(t)}.$$

(ii) If t is right-dense, then g is differentiable at t if and only if the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case,

$$g^\Delta(t) = \lim_{s\to t} \frac{g(t) - g(s)}{t - s}.$$

Now, we consider some particular solutions to (1.1). Let $x(t)$ be a solution of (1.1). First, suppose that $x(t) > 0$ for $t \in [2k - 2, 2k - 1]$ and $k \in \mathbb{N}$. Then it follows from (1.1) that

$$x^\Delta(t) = -\frac{1}{2}x(t) - 1, \quad \text{for } t \in [2k, 2k + 1], \tag{2.1}$$

and hence

$$x(t) = e^{-\frac{1}{2}(t-2k)} \left[x(2k) - 2\left(e^{\frac{1}{2}(t-2k)} - 1\right)\right], \quad t \in (2k, 2k + 1]. \tag{2.2}$$

Second, suppose that $x(t) \le 0$ for $t \in [2k - 2, 2k - 1]$ and $k \in \mathbb{N}$. Similarly, we have

$$x(t) = e^{-\frac{1}{2}(t-2k)} \left[x(2k) + 2\left(e^{\frac{1}{2}(t-2k)} - 1\right)\right], \quad \text{for } t \in (2k, 2k + 1]. \tag{2.3}$$

Assume that $k \ge 2$, if $x(2k - 3) > 0$, then

$$x(2k) = \frac{1}{2}x(2k - 1) + f(x(2k - 3)) = \frac{1}{2}x(2k - 1) - 1; \tag{2.4}$$

otherwise,

$$x(2k) = \frac{1}{2}x(2k - 1) + f(x(2k - 3)) = \frac{1}{2}x(2k - 1) + 1. \tag{2.5}$$

We conclude this section with the definition of stability.

Definition 2.1. The solution $x_{\Phi_0}(t)$ of (1.1) is called stable if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\Phi(t) - \Phi_0(t)| < \delta$ for $t \in [0, 2]_{\mathbb{T}}$, then we have $|x_\Phi(t) - x_{\Phi_0}(t)| < \varepsilon$ for $t \in (2, \infty)_{\mathbb{T}} = (2, \infty) \cap \mathbb{T}$.

Definition 2.2. The solution $x_{\Phi_0}(t)$ of (1.1) is called asymptotically stable if it is stable and if there exists a $\eta > 0$ such that $|\Phi(t) - \Phi_0(t)| < \eta$ for $t \in [0, 2]_{\mathbb{T}}$, then we have $\lim_{t\to\infty} |x_{\Phi}(t) - x_{\Phi_0}(t)| = 0$.

3. Main results

Now, we are ready to establish the main results of this paper.

Theorem 3.1. *Eq. (1.1) has an 8-periodic solution $x_{\bar{\Phi}}(t)$ satisfying $\bar{\Phi}(t) > 0$ for $t \in [0, 2]_{\mathbb{T}}$.*

Proof. Choose $\Phi_0 : [0, 2]_{\mathbb{T}} \to \mathbb{R}$ such that $\Phi_0(t) > 0$ for $t \in [0, 1]$ and

$$\Phi_0(2) = \frac{2e^{-2} - 16e^{-1} + 32}{16 - e^{-2}} > 0. \tag{3.1}$$

Then, from (2.2) and (3.1), we get

$$x_{\Phi_0}(3) = e^{-\frac{1}{2}}\left[\Phi_0(2) - 2\left(e^{\frac{1}{2}} - 1\right)\right] = \frac{2e^{-2} - 16e^{-\frac{3}{2}} + 64e^{-\frac{1}{2}} - 32}{16 - e^{-2}} > 0,$$

$$x_{\Phi_0}(4) = \frac{1}{2}x_{\Phi_0}(3) - 1 = \frac{2e^{-2} - 8e^{-\frac{3}{2}} + 32e^{-\frac{1}{2}} - 32}{16 - e^{-2}} \in (-1, 0),$$

and hence, for $t \in (4, 5]$,

$$x_{\Phi_0}(t) = e^{-\frac{1}{2}(t-4)}\left[x_{\Phi_0}(4) - 2\left(e^{\frac{1}{2}(t-4)} - 1\right)\right] < x_{\Phi_0}(4) < 0.$$

Especially,

$$x_{\Phi_0}(5) = e^{-\frac{1}{2}}\left[x_{\Phi_0}(4) - 2\left(e^{\frac{1}{2}} - 1\right)\right] = \frac{-6e^{-2} + 32e^{-1} - 32}{16 - e^{-2}} < 0,$$

$$x_{\Phi_0}(6) = \frac{1}{2}x_{\Phi_0}(5) - 1 = \frac{-2e^{-2} + 16e^{-1} - 32}{16 - e^{-2}} < 0,$$

and by (2.3), for $t \in (6, 7]$

$$x_{\Phi_0}(t) \le x_{\Phi_0}(7) = \frac{16e^{-\frac{3}{2}} - 64e^{-\frac{1}{2}} - 2e^{-2} + 32}{16 - e^{-2}} < 0.$$

Similarly, we have

$$x_{\Phi_0}(8) = \frac{1}{2}x_{\Phi_0}(7) + 1 = \frac{-2e^{-2} + 8e^{-\frac{3}{2}} - 32e^{-\frac{1}{2}} + 32}{16 - e^{-2}} > 0,$$

$$x_{\Phi_0}(t) = e^{-\frac{1}{2}(t-8)}x_{\Phi_0}(8) + 2\left(1 - e^{-\frac{1}{2}(t-8)}\right) > 0,$$

for $t \in (8, 9]$. From (2.5) and (3.1), we get

$$x_{\Phi_0}(10) = \frac{1}{2}x_{\Phi_0}(9) + 1 = \frac{2e^{-2} - 16e^{-1} + 32}{16 - e^{-2}} = \Phi_0(2).$$

Therefore, for $t \in [10, 11]$

$$x_{\Phi_0}(t) = e^{-\frac{1}{2}(t-10)}\left[\Phi_0(2) - 2\left(e^{\frac{1}{2}(t-10)} - 1\right)\right]. \tag{3.2}$$

Repeating this procedure, we can obtain $x_{\Phi_0}(t) = x_{\Phi_0}(t + 8)$ for $t \in [2, \infty)_{\mathbb{T}}$. Let

$$\bar{\Phi}(t) = x_{\Phi_0}(t + 8), \quad \text{for } t \in [0, 2]_{\mathbb{T}},$$

then $x_{\bar{\Phi}}(t)$ is an 8-periodic solution of (1.1). The proof is completed. □

Theorem 3.2. *Eq. (1.1) has an 8-periodic solution $x_{\Phi^*}(t)$ satisfying $\Phi^*(t) < 0$ for $t \in [0, 2]_{\mathbb{T}}$.*

The proof of Theorem 3.2 is similar to that of Theorem 3.1, we omit it.

Theorem 3.3. *Assume that $k \in \mathbb{N}$. Let $x_{\Phi}(t)$ be a solution of (1.1) satisfying the initial condition $\Phi(t) > 0$ for $t \in [0, 2]_{\mathbb{T}}$ and $\Phi(2) \in (2^k e^{\frac{k}{2}} - 2, 2^{k+1}e^{\frac{k}{2}} - 2]$. Then $x_{\Phi}(t + 2k - 2) \to x_{\bar{\Phi}}(t)$ as $t \to \infty$. Moreover, $x_{\bar{\Phi}}(t)$ is asymptotically stable; Let $x_{\Phi}(t)$ be a solution of (1.1) satisfying the initial condition $\Phi(t) < 0$ for $t \in [0, 2]_{\mathbb{T}}$ and $\Phi(2) \in (2 - 2^{k+1}e^{\frac{k}{2}}, 2 - 2^k e^{\frac{k}{2}}]$. Then $x_{\Phi}(t + 2k - 2) \to x_{\Phi^*}(t)$ as $t \to \infty$. Moreover, $x_{\Phi^*}(t)$ is asymptotically stable.*

Proof. We will prove the first part, and the second part is similar and the proof is omitted.

First, we will prove that $x_\Phi(t+2k-2) \to x_{\bar\Phi}(t)$ as $t\to\infty$ by induction.

If the conclusion holds for $k=1$, then $\Phi(t)>0$ for $t\in[0,1]$ and $\Phi(2)\in(2e^{\frac12}-2, 4e^{\frac12}-2]$. In view of (2.2), we have

$$x_\Phi(t) = e^{-\frac12(t-2)}\left[\Phi(2) - 2\left(e^{\frac12(t-2)}-1\right)\right] \ge x_\Phi(3) = e^{-\frac12}\Phi(2) - 2\left(1-e^{-\frac12}\right) > 0,$$

for $t\in[2,3]$. Thus,

$$x_\Phi(4) = \frac12 x_\Phi(3) + f(x_\Phi(1)) = \frac12 x_\Phi(3) - 1 = \frac12 e^{-\frac12}\Phi(2) + e^{-\frac12} - 2 \in (-2, 0],$$

$$x_\Phi(t) = e^{-\frac12(t-4)}x_\Phi(4) - 2\left(1-e^{-\frac12(t-4)}\right) < x_\Phi(4) \le 0,$$

for $t\in(4,5]$. Now,

$$x_\Phi(6) = \frac12 x_\Phi(5) - 1 < 0,$$

$$x_\Phi(t) \le x_\Phi(7) = e^{-\frac12}x_\Phi(6) + 2\left(1-e^{-\frac12}\right) = e^{-1} - 4e^{-\frac12} + 2 < 0,$$

for $t\in[6,7]$, Therefore,

$$x_\Phi(8) = \frac12 x_\Phi(7) + 1 > \frac18 e^{-\frac32}\left(2e^{\frac12}-2\right) + \frac14 e^{-\frac32} - 2e^{-\frac12} + 2 = \frac14 e^{-1} - 2e^{-\frac12} + 2 > 0,$$

$$x_\Phi(t) = e^{-\frac12(t-8)}x_\Phi(8) + 2\left(1-e^{-\frac12(t-8)}\right) > 0,$$

for $t\in[8,9]$. Similarly,

$$x_\Phi(10) = \frac12 x_\Phi(9) + 1 = \frac1{16}e^{-2}\Phi(2) + \frac18 e^{-2} - e^{-1} + 2.$$

This leads to $x_\Phi(10)\in\left(\frac18 e^{-\frac32} - e^{-1} + 2, \frac14 e^{-\frac32} - e^{-1} + 2\right] \subset \left(2e^{\frac12}-2, 4e^{\frac12}-2\right]$. By iteration, we get

$$\begin{cases} x_\Phi(t) > 0, & \text{if } t\in[8n, 8n+3]_{\mathbb{T}},\\ x_\Phi(t) < 0, & \text{if } t\in[8n+4, 8n+7]_{\mathbb{T}},\end{cases} \tag{3.3}$$

where $n\in\mathbb{N}_0$. This implies that $x_\Phi(t)$ and $x_{\bar\Phi}(t)$ have the same sign on $\mathbb{T}$. Therefore,

$$x_\Phi(t) - x_{\bar\Phi}(t) = \left(\frac12\right)^n e^{-\frac n2}e^{-\frac12(t-2n-2)}\left(\Phi(2)-\bar\Phi(2)\right), \quad \text{for } t\in[2n, 2n+1], \tag{3.4}$$

where $n\in\mathbb{N}$. Clearly, $x_\Phi(t)\to x_{\bar\Phi}(t)$ as $t\to\infty$.

If the conclusion holds for $k=m$, that is $\Phi(t)>0$ for $t\in[0,2]_{\mathbb{T}}$ and $\Phi(2)\in\left(2^m e^{\frac m2}-2, 2^{m+1}e^{\frac m2}-2\right]$, then $x_\Phi(t+2m-2)\to x_{\bar\Phi}(t)$ as $t\to\infty$.

We consider the case where $k=m+1$, then $\Phi(t)>0$ for $t\in[0,2]_{\mathbb{T}}$ and $\Phi(2)\in\left(2^{m+1}e^{\frac{m+1}2}-2, 2^{m+2}e^{\frac{m+1}2}-2\right]$. By (2.2), for $t\in(2,3]$,

$$x_\Phi(t) = e^{-\frac12(t-2)}\left[\Phi(2) - 2\left(e^{\frac12(t-2)}-1\right)\right] \ge x_\Phi(3) = e^{-\frac12}\Phi(2) - 2\left(1-e^{-\frac12}\right).$$

By (2.4),

$$x_\Phi(4) = \frac12 x_\Phi(3) + f(x_\Phi(1)) = \frac12 x_\Phi(3) - 1 = \frac12 e^{-\frac12}\Phi(2) + e^{-\frac12} - 2.$$

This implies that $x_\Phi(4)\in\left(2^m e^{\frac m2}-2, 2^{m+1}e^{\frac m2}-2\right]$ and $x_\Phi(t)>0$ for $t\in[2,4]_{\mathbb{T}}$.

Let $\Psi(t) = x_\Phi(t+2)$ for $t\in[0,2]_{\mathbb{T}}$, we have $x_\Psi(t) = x_\Phi(t+2)$ for $t\in(2,\infty)_{\mathbb{T}}$, and $x_\Psi(t+2m-2)\to x_{\bar\Phi}(t)$ as $t\to\infty$, that is $x_\Phi(t+2m)\to x_{\bar\Phi}(t)$ as $t\to\infty$. Thus, the conclusion holds for $k=m+1$.

Next, we will show that the periodic solution $x_{\bar\Phi}(t)$ is asymptotically stable.

For any $\varepsilon>0$, there is no harm in assuming that $\varepsilon<\min\left\{\bar\Phi(t): t\in[0,2]_{\mathbb{T}}\right\}$. Let $\delta=\min\{\varepsilon, \bar\Phi(2)-2e^{\frac12}+2, 4e^{\frac12}-2-\bar\Phi(2)\}$, $|\Phi(t)-\bar\Phi(t)|<\delta$ for $t\in[0,2]_{\mathbb{T}}$ implies that $\Phi(t)>0$ for $t\in[0,2]_{\mathbb{T}}$ and $\Phi(2)\in\left(2e^{\frac12}-2, 4e^{\frac12}-2\right)$. Therefore, (3.4) holds and

$$|x_\Phi(t) - x_{\bar\Phi}(t)| < \varepsilon$$

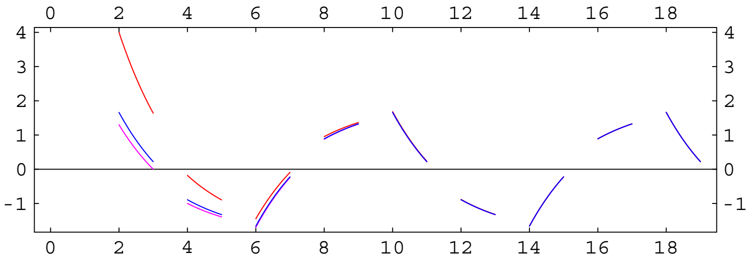

Fig. 1. The stability of $x_{\bar{\Phi}}(t)$.

holds for $t \in (2, \infty)_{\mathbb{T}}$. From (3.4), we have also

$$\lim_{t\to\infty} |x_{\Phi}(t) - x_{\bar{\Phi}}(t)| = 0.$$

Therefore, $x_{\bar{\Phi}}(t)$ is asymptotically stable. The proof is completed. □

Let us conclude this paper with some numerical simulations. In Fig. 1, the upper, the middle, and the lower curves corresponding to the solutions of (1.1) satisfying the initial conditions $\Phi(t) > 0$ for $t \in [0, 1]$ and $\Phi(2) = 4, 1.66(\bar{\Phi}(2))$ and 1.3, respectively. It is clear that the upper and the lower curves approach the middle one as t increases.

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